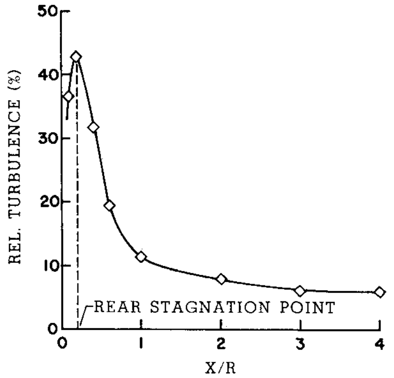

Fig. 6 Centerline relative turbulence intensity.

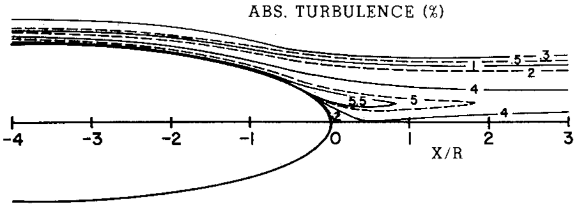

Fig. 7 Absolute turbulence intensity map.

proximately $Y/R = 0.1$ below the point of maximum shear stress.

As mentioned previously, the relative turbulence intensity can provide some useful information. On a relative scale, turbulence levels as high as 60% were observed around the separation bubble; on an absolute scale, the turbulence level was only 6%. Care must be exercised when quoting these large values of relative turbulence since they occur in regions where the local mean velocity is small.[6] A plot of the centerline relative turbulence level against axial distance downstream from the model tip is shown in Fig. 6. A peak in the relative turbulence is found at $X/R = 0.2$, the rear stagnation point as determined by other methods.[4] Downstream of the rear stagnation point the relative turbulence intensity declines rapidly due to the increase in centerline velocity. The peak in the relative turbulence curve can be used to indicate the rear stagnation point while a look at Fig. 4 shows that the absolute turbulence intensity cannot be used to find the rear stagnation point.

Combining the results in the near-wake with the data of Yi and Przirembel[3] for the approaching boundary layer on the model, a map of the absolute turbulence intensity may be constructed for the entire flowfield. This is shown in Fig. 7. It can be seen that the absolute turbulence intensity in the boundary layer remains nearly constant along the streamlines. This is also true in the outer edges of the free shear layer. A turbulent core is formed in the central part of the free shear layer. Within the separation bubble, the turbulence is small.

Conclusions

The turbulent near-wake of a semielliptical afterbody has been investigated experimentally using a hot-wire anemometer system. A complete map of the absolute turbulence intensity has been obtained. It shows a maximum turbulence level of approximately 6% in a small core region just adjacent to the separation bubble. On the near-wake centerline, the maximum absolute turbulence intensity was 4%; it occurred just downstream of the rear stagnation point. The absolute turbulence in the wake decays slowly, persisting to distances well in excess of 10 model radii downstream of the model tip. At all axial locations in the near-wake, the point of maximum turbulence was located $Y/R = 0.1$ below the point of maximum shear stress in each velocity profile.

On a relative scale, the turbulence intensity is much larger throughout the near-wake than on an absolute scale. Turbulent fluctuations in excess of 60% of the local mean velocity were observed in and around the separation bubble. The relative turbulence intensity can be used to indicate the presence of a stagnation point. It successfully located the rear stagnation point in the near-wake at $X/R = 0.2$.

The information presented in this paper on turbulence intensities, when coupled with the previous work of Yi and Przirembel[3] and Merz et al.[4] provides a complete description of the flowfield about a semielliptical afterbody. The results will be useful in developing and evaluating analytical solutions for the flowfield.

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Incremental Multigrid Strategy for the Fluid Dynamic Equations

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I. Introduction

OVER the past few years, the author has developed implicit schemes for the numerical solution of the Navier-Stokes and lambda-formulation Euler equations, mostly for the case of steady flows. The basic framework that underlies the development of these numerical methods[1-4] follows. The governing equations are discretized in time by a two-level implicit Euler time stepping and linearized using the Taylor series and the incremental (delta) approach of Beam and Warming.[5] The large, block-pentadiagonal, linear system that results at every time step is reduced to a series of smaller block-tridiagonal systems by approximate factorization[1-3] or simple directional mutilation.[4] These smaller systems are solved very efficiently by means of standard block-tridiagonal Gaussian elimination.[6] The solution is then updated and the process is repeated until a satisfactory convergence criterion is met.

A positive feature of this approach is that a deferred-correction strategy[7] can be implemented in a very easy and

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elegant manner by virtue of the delta form of the equations.[3] First-order-accurate upwind differences are used for the advection terms in the left-hand side (LHS) implicit operator, whereas second-order-accurate central[3,4] or upwind[1,2] differences are used to explicitly evaluate the right-hand side (RHS), which turns out to be the steady-state equation. In this way, the linear systems that will be solved at every time step are guaranteed to be diagonally dominant, thereby giving the iteration process improved stability and convergence rate while preserving the accuracy of the final steady-state solution. However, when refining the mesh, the convergence rate of all of these methods invariably deteriorates so that, for complicated problems requiring a very fine computational mesh, they tend to lose their competitiveness. It therefore appears timely and worthwhile to remove such a limitation by developing a new and simple multigrid strategy especially designed for this kind of incremental method.

II. Methodology

The proposed incremental multigrid strategy is described for the particular case of the vorticity-stream function Navier-Stokes equations

$$\omega_t + \psi_y \omega_x - \psi_x \omega_y - (\omega_{xx} + \omega_{yy})/Re = 0 \tag{1}$$

$$\psi_t - \psi_{xx} - \psi_{yy} - \omega = 0 \tag{2}$$

In Eqs. (1) and (2) Re is the Reynolds number, ω is the vorticity, ψ is the stream function, x and y are the standard Cartesian coordinates, t is the time, the subscripts indicate partial derivatives, and a relaxation-like time derivative has been added to the stream function equation to parabolize it. Equations (1) and (2) are discretized and linearized in time using a two level implicit Euler scheme and the delta form[5] to give

$$\begin{aligned}
&\Delta\omega^H/\Delta t + \psi_y^n \Delta\omega_x^H + \omega_x^n \Delta\psi_y^H - \psi_x^n \Delta\omega_y^H - \omega_y^n \Delta\psi_x^H \\
&\quad - (\Delta\omega_{xx}^H - \Delta\omega_{yy}^H)/Re = C_h^H[-(\psi_y^n\omega^n)_x + (\psi_x^n\omega^n)_y \\
&\quad + (\omega_{xx}^n + \omega_{yy}^n)/Re] = C_h^H\,\mathrm{RES}(\omega)^n
\end{aligned} \tag{3}$$

$$\begin{aligned}
&\Delta\psi^H/\Delta t - \Delta\psi_{xx}^H - \Delta\psi_{yy}^H - \Delta\omega = C_h^H[\psi_{xx}^n + \psi_{yy}^n + \omega] \\
&\quad = C_h^H \mathrm{RES}(\psi)^n
\end{aligned} \tag{4}$$

In Eqs. (3) and (4) Δt is the time step, $\Delta\omega = \omega^{n+1} - \omega^n$ (the superscripts n and $n+1$ indicating the old and new time levels t^n and $t^{n+1} = t^n + \Delta t$) etc., the superscript H indicates the current grid and C_h^H indicates a suitable collection operator from the finest grid h to the current grid H. The superscript h, indicating the finest grid h is always omitted, for convenience. Notice in passing that the conservative form of the convection terms is used in the RHS of Eq. (3) since it has been shown to provide improved accuracy over the advection form, especially at high Re values (see, e.g., Ref. 4).

Equations (3) and (4) are discretized in space using second-order-accurate central differences throughout, except for the incremental convective terms of the vorticity equation in the LHS of Eq. (3), which are approximated using first-order-accurate upwind differences. The resulting 2×2 block-pentadiagonal system is then approximately solved by a single-sweep alternating direction block-line-Gauss-Seidel method[4] that solves only block-tridiagonal systems. The solution process is started on the finest grid, with $\Delta\omega^H = \Delta\omega$, $\Delta\psi^H = \Delta\psi$. The solution ($\omega^n$, ψ^n) is updated and Eqs. (3) and (4) are then solved on successively coarser grids ($H = 2h$, $4h, \ldots$) until the finest-grid residual is reduced to a suitably small value. In more detail, at every grid level H, the following steps are required by the proposed multigrid strategy: 1) the coefficients of Eqs. (3) and (4) are evaluated from the finest-grid (updated) solution ω^n, ψ^n locally at the H mesh gridpoints; 2) the RHS of Eqs. (3) and (4) are evaluated from the finest grid and collected up to the current grid H; 3) Eqs. (3) and (4) are approximately solved using a single sweep of the aforementioned smoother and homogeneous Dirichlet boundary conditions to provide $\Delta\omega^H$, $\Delta\psi^H$; 4) $\Delta\omega$, $\Delta\psi$ are evaluated as

$$(\Delta\omega, \Delta\psi) = I_H^h(\Delta\omega^H, \Delta\psi^H) \tag{5}$$

where I_H^h is the standard bilinear interpolation operator from the current grid H to the finest grid h; 5) the (finest-grid) solution is finally updated as

$$(\omega^n, \psi^n) \leftarrow (\omega^n, \psi^n) + (\Delta\omega, \alpha\Delta\psi) \tag{6}$$

where α is a suitable overrelaxation factor; and 6) the (finest-grid) vorticity at the wall is recomputed using the correct

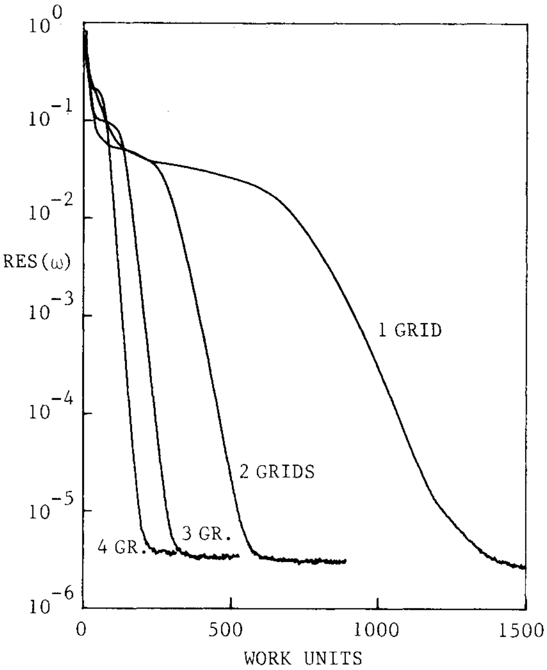

Fig. 1 Convergence history for the Navier-Stokes equations.

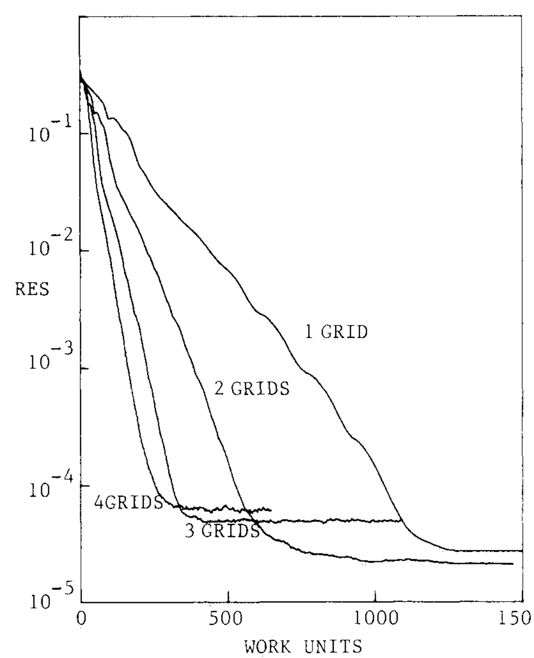

Fig. 2 Convergence history for the lambda-formulation Euler equations.

Neumann boundary condition for the stream function, a point-image and the steady-state stream function equation evaluated at the wall.

It is noteworthy that the proposed approach is applicable to any set of time-discretized systems of partial differential equations. Moreover, it is extremely simple since it does not require that any "logical choices" be made or "free parameters" be tuned and it does not need any additional storage with respect to the basic smoother (insofar as only the finest-grid solution is computed and a single array is used for the deltas at all grid levels). However, its work per iteration is slightly greater than that required by most current multigrid methods, additional interpolations and collections being needed to visit and update the finest-grid solution after every coarse-grid calculation. Also, no convergence theorem exists for this method, so that numerical experiments are required to assess its merits.

III. Results

The proposed methodology has been applied to solve three different sets of equations, namely, the Laplace equation, the vorticity-stream function Navier-Stokes equations (for the case of the classical driven cavity flow at $Re = 1000$), and the lambda-formulation Euler equations (for the case of a two-dimensional subsonic source-flow). For the first, simple problem, the method possesses a typical multigrid convergence rate that is practically independent of the size of the finest grid used in the computation (see Ref. 8 for details). For the second, reasonably difficult problem, the method, as previously described, was applied starting from rest, using a nonoptimized unitary time step for both equations, an overrelaxation factor $\alpha = 1.5$, and the standard 9-point collection operator for the residual. The convergence history of the method, using a 65×65 uniform finest mesh and from one to four grid levels, is given in Fig. 1 as the average residual of the vorticity equation versus the work units. The improvement provided by the proposed multigrid approach is quite clear—machine zero (using single precision arithmetic on an HP 9000/9050 minicomputer) is obtained after almost 1500 work units (iterations) using the single grid smoother and after only about 200 work units using four grid levels. It appears quite remarkable, indeed, that such a naive method not only improves the asymptotic convergence rate significantly but also reduces the length of the plateau in the convergence history. It may be of interest to know that the work units per iteration, for the present calculation, are equal to 1.49, 1.88, and 2.26 for the case of two, three, and four grids, respectively (one work unit requires about 8 CPU seconds on the aforementioned minicomputer).

The proposed approach was finally applied to solve the lambda-formulation Euler equations in two dimensions for the case of a subsonic source flow. For such a case, an explicit one-step method was used as smoother, simple injection was used also for the RHS residual and the time step, limited by the CFL condition, was doubled on each coarser-mesh computation. The convergence rate is given in Fig. 2 as the average residual of the three governing equations versus the work units. The validity of the proposed multigrid strategy is again quite clear. It is noteworthy that machine zero is reached at a higher value of the residual (which is proportional to the computed deltas divided by the time step) and is due to the small time step used in the calculations. For the present computations, using an explicit smoother and simple injection for the residuals, the work units per step are equal to 1.47, 1.82, and 2.16 for the case of two, three, and four grid levels, respectively (one work unit requires about 2 CPU seconds).

IV. Conclusions and Future Work

A novel incremental multigrid strategy has been proposed to solve the equations of fluid dynamics. It provides considerable efficiency gains for the case of elliptic, mixed parabolic-elliptic, and hyperbolic problems. With respect to current well-established multigrid schemes, the proposed approach is algorithmically simpler but requires more work per iteration and, at present, has been shown to be effective only for subsonic flow problems and uniform grids. Current work is attempting to remove both limitations and to demonstrate the applicability of the method to the very important case of the compressible Navier-Stokes equations in conservation-law form.

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Wavy Wall Solutions of the Euler Equations

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Introduction

THE steady inviscid irrotational flow of a compressible fluid past a small amplitude sinusoidally varying wavy wall has been well explained using small perturbation theory.[1] For subsonic and supersonic flow, the solutions are analytically obtained by neglecting higher order terms in the pertur-

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